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MAXIMUM SET OF EDGES NO TWO COVERED BY A CLIQUE

A. V. KOSTOCHKA

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Let $h(G)$ be the largest number of edges of the graph G, no two of which are contained in the same clique. For G without isolated vertices it is proved that if $h(G)\leqq 5$, then $\chi(\bar{G})\leqq h(G)$, but if $h(G)=6$ then $\chi(\bar{G})$ can be arbitrarily large.

1. Introduction and notations

The minimal number of cliques which cover all edges of a graph G, denoted by $cc(G)$, equals the minimal cardinality of such a set S that G is the intersection graph of some system of subsets of S [3, 5]. The natural lower bound for $cc(G)$ is the magnitude $h(G)$ — the largest number l such that there exists a subset of edges $E'=\{e_1, e_2, \dots, e_l\}$ of the graph G possessing the property:

(1) *Any clique in G contains no more than one edge from E'.*

Parthasaraty and Choudum [6] conjectured that for any graph G without isolated vertices holds

(2) $\chi(\bar{G})\leqq h(G)$,

where $\bar{G}$ is the complement of the graph G and $\chi(\bar{G})$ is its chromatic number. Brigham and Dutton [1] proved that (2) is valid for graphs G with $h(G)\leqq 2$. Erdős [2] proved that (2) is false for almost all graphs with n vertices and, moreover, for every sufficiently large t and n $\big(t\geqq t_0, n\geqq n_0(t)\big)$ there exists a graph $G_{n,t}$ with n vertices having the following properties:

$$(3)\qquad \begin{cases} \text{(i)} & h(G_{n,t}) \leqq t, \\ \text{(ii)} & \chi(\bar{G}_{n,t}) > n^{c_t}, \text{ where } c_t \text{ does not depend on } n, \\ \text{(iii)} & G_{n,t} \text{ has no isolated vertices.} \end{cases}$$

In the same paper Erdős asked:

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1) What is the maximal t_1 such that for any graph G without isolated vertices with $h(G) \leqq t_1$ (2) holds?

2) Find the least t_0, for which (3) is valid, and the best values for c_t.

The goal of the present paper is to prove that $t_1 = 5$, $t_0 = 6$ and $c_6 > 1/15$.

It is interesting that for graphs G without isolated vertices from $h(G) = 5$ there follows (2), and if $h(G) = 6$ then $\chi(\bar{G})$ can be arbitrarily large.

We shall use the following notations. If $G = (V, E)$ then $\bar{G} = (V, \bar{E})$ is the complement of the graph G, $g(G)$ is the girth of the graph G, $N_G(v) = \{w \in V \setminus \{v\} | (v, w) \in E\}$, $s_G(v) = |N_G(v)|$.

A subset E' of the graph's edges, satisfying (1), is called a sparse set.

2. The lower bound for t_1

Proposition 1. *If $\chi(G) \leqq 6$ and $\bar{G}$ has no isolated vertices then $\chi(G) \leqq h(\bar{G})$.*

Proof. Let $G = (V, E)$ be the least counterexample to the proposition, $\bar{G} = (V, \bar{E})$. Then

$$6 \geqq \chi(G) \geqq h(\bar{G}) + 1. \tag{4}$$

It is obvious that $|V| \geqq 4$. Let $(v, u) \in E$. Since $\bar{G}$ has no isolated vertices then there exist $w, z \in V$ such that $\{(w, v), (u, z)\} \subset \bar{E}$. Consequently, $h(\bar{G}) \geqq 2$ and $\chi(G) \geqq 3$.

Next we obtain some properties of the graph G.

(A) *For any $w \in V$ the subgraph of the graph $\bar{G}$, generated by $N_{\bar{G}}(w)$, is not complete.*

Proof of (A). Assume the contrary. Choose a vertex v_1 with the minimal degree in $\bar{G}$ among the vertices w for which the subgraph of the graph $\bar{G}$ on $N_{\bar{G}}(w)$ is complete. Consider $\bar{G}_1 = \bar{G} \setminus N_{\bar{G}}(v_1)$. Let just the vertices of the set $V_1 = \{v_1, v_2, \ldots, v_t\}$ be isolated in $\bar{G}_1$. If $v_i \in V_1$ and $i > 1$ then, due to the choice of v_1, we have $N_{\bar{G}}(v_i) = N_{\bar{G}}(v_1)$. Put $\bar{G}_2 = \bar{G}_1 \setminus V_1$. By construction, $\bar{G}_2$ has no isolated vertices. Since the subgraph of $\bar{G}$ on vertices $N_{\bar{G}}(v_1)$ is complete then $\chi(G_2) \geqq \chi(G) - |V_1| = \chi(G) - t$. By the inductive assumption $\chi(G_2) \leqq h(\bar{G}_2)$. Let $\{\bar{e}_1, \ldots, \bar{e}_l\}$ be a sparse set in $\bar{G}_2$ and $l = h(\bar{G}_2)$. Since $\bar{G}$ has no isolated vertices then one can choose for $i = 1, 2, \ldots, t$ the edge $\bar{e}_{l+i} \in \bar{E}$ incident to v_i. Since V_1 is an independent set in $\bar{G}$ then the edges $\bar{e}_{l+i}$ and $\bar{e}_{l+j}$ for $i \neq j$ do not belong to the same clique. Finally, for any $1 \leqq s \leqq l$ both ends of $\bar{e}_s$ are not incident in $\bar{G}$ to vertices from V_1. Consequently, $h(\bar{G}) \geqq l + t = h(\bar{G}_2) + t \geqq (\chi(G) - t) + t = \chi(G)$, which contradicts to (4). ∎

(B) *G is colour-critical.*

Proof of (B). Let $v \in V$. Consider $G \setminus v$. According to (A), there are no vertices of degree 1 in $\bar{G}$. So, $\bar{G} \setminus v$ has no isolated vertices and, due to inductive assumption, $\chi(G \setminus v) \leqq h(\bar{G} \setminus v)$. But $h(\bar{G} \setminus v) \leqq h(\bar{G}) < \chi(G)$. ∎

(C) *There are no 3-cycles in G.*

Proof of (C). Suppose that (v_1, v_2, v_3) is a 3-cycle in G. According to (A), for every $i \in \{1, 2, 3\}$ there exists such an edge $(w_i, u_i) \in E$ such that $\{w_i, u_i\} \subset N_{\bar{G}}(v_i)$. Then the set $\{(v_i, w_i), (v_i, u_i) | i=1, 2, 3\}$ is sparse. Consequently, $h(\bar{G}) \geqq 6$. But $\chi(G) \leqq 6$. ▮

(D) *If* $(u, v) \notin E$ *then there exists a vertex* $w \in V$ *such that* $(u, w) \in E$, $(v, w) \notin E$.

Proof of (D). If $N_G(u) \subset N_G(v)$ then $\chi(G \setminus u) = \chi(G)$, which contradicts to (B). ▮

(E) *For any chain* $(v_1, v_2, \ldots, v_6)$ *of length 5 in G*

$$\{(v_1, v_5), (v_1, v_6), (v_2, v_6), (v_2, v_5)\} \cap E \neq \emptyset. \tag{5}$$

Proof of (E). We show that if for some chain $(v_1, v_2, \ldots, v_6)$ condition (5) does not hold then $h(\bar{G}) \geqq 6$ contrary to (4). According to (C), $(v_1, v_3) \notin E$, $(v_4, v_6) \notin E$. According to (D), there exists a $v_7 \in V$ such that $(v_6, v_7) \in E$, $(v_4, v_7) \notin E$. Since $\{v_3, v_5\} \subset N_G(v_4)$, $v_1, v_2 \notin N_G(v_6)$ then $v_7 \notin \{v_1, v_2, \ldots, v_6\}$. Similarly, there is such a vertex $v_8 \notin \{v_1, v_2, \ldots, v_6\}$ that $(v_1, v_8) \in E$, $(v_3, v_8) \notin E$ (perhaps, $v_7 = v_8$). Then the set $\{(v_1, v_5), (v_1, v_6), (v_2, v_5), (v_2, v_6), (v_3, v_8), (v_4, v_7)\}$ is sparse (see Fig. 1). Consequently, $h(\bar{G}) \geqq 6$. ▮

From (E) immediately follows

(F) *Every k-cycle in G for* $k \geqq 7$ *has a chord.* ▮

(H) $\chi(G) = 6$.

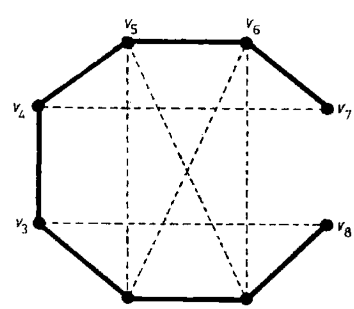

Fig. 1

Proof of (H). Since $\chi(G) \geqq 3$ then in G there is an odd cycle without chord. According to (C) and (F), this is a 5-cycle $(w_1, w_2, w_3, w_4, w_5)$. But then $\{(w_i, w_{i+2}) | i = 1, 2, 3, 4, 5\}$ (addition modulo 5) is a sparse set in $\bar{E}$ and $h(\bar{G}) \geqq 5$. According to (4), $5+1 \leqq \chi(G) \leqq 6$. ▮

Completion of the proof of Proposition 1. Let $(v_1, v_2) \in E$, $V_1 = N_G(v_1)$, $V_2 = N_G(v_2)$, $G_0 = G \setminus (V_1 \cup V_2)$. According to (C), V_1 and V_2 are independent sets. Consequently, $\chi(G) \leqq \chi(G_0) + 2$. So, in view of (H), $\chi(G_0) \geqq 4$ and in G_0 there is an odd cycle $C = (x_1, x_2, \ldots, x_{2t+1})$ $(t \geqq 2)$. Consider the shortest chain $(u_1, u_2, \ldots, u_k)$ connecting in G the cycle C with $\{v_1, v_2\}$. One can assume that $u_1 = x_1$, $u_k = v_1$. Since $C \cap (V_1 \cup V_2) = \emptyset$ then $k \geqq 3$ and $(v_2, u_i) \notin E (1 \leqq i \leqq k-1)$. Then any segment of length 5 of the chain $(x_3, x_2, u_1, u_2, \ldots, u_k, v_2)$ contradicts to the Claim (E). ▮

3. Sparse sets in complements of graphs with girth 8

It is easy to prove by induction

Lemma 1. *If* $G=(V, E)$, $|E|\geqq|V|+1$, *then*

$$g(G) \leqq \frac{2}{3}(|V|+1). \quad \blacksquare$$

Lemma 2. *Let* $G_0=(V_0, E_0)$, $|V_0|=10$ *and* $\bar{\pi}=\{\bar{e}_i=(v_{1i}, v_{2i})|i=1,2,3,4,5\}$ *be a sparse matching in* $\bar{G}_0$. *Then either* G_0 *is a* 10*-cycle, or* $g(G_0)\leqq 7$.

Since the statement of Lemma 2 is a very special fact, we only sketch the proof. Suppose that the lemma is wrong for G_0.

1. Since $\bar{\pi}$ is a sparse matching the sets $\{v_{1i}, v_{2i}\}$ and $\{v_{1j}, v_{2j}\}$ for $i\neq j$ are connected in G_0 by at least one edge. Consequently, for every $1\leqq i\leqq 5$

$$s_{G_0}(v_{1i})+s_{G_0}(v_{2i}) \geqq 4 \tag{6}$$

and $|E_0|\geqq 10$.

2. If $|E_0|\geqq 11$ then, by Lemma 1, $g(G_0)\leqq 7$. Consequently, $|E_0|=10$, and in all the relations of (6) equality holds.

3. If for some vertex $v\in V_0$ $s_{G_0}(v)=0$ then the graph $G_0\setminus v$ has 9 vertices and 10 edges. Then, by to Lemma 1, $g(G_0\setminus v)\leqq 6$ holds. So, for every vertex $v\in V_0$ we have $1\leqq s_{G_0}(v)\leqq 3$.

4. If G_0 is a 2-regular graph and not a 10-cycle then $g(G_0)\leqq 5$.

5. Let $s_{G_0}(v_{11})=1$, $(v_{11}, v_{12})\in E_0$. Then $s_{G_0}(v_{21})=3$ and $(v_{21}, v_{12})\notin E_0$. Put $G_1=(V_1, E_1)=G_0\setminus\{v_{11}, v_{12}\}$. If $s_{G_0}(v_{12})\leqq 2$ then we have $|V_1|=8$, $|E_1|\geqq 8$. Besides, $s_{G_1}(v_{21})=3$. Consequently, $g(G_1)\leqq 7$.

6. Thus, $s_{G_0}(v_{12})=3$, $s_{G_0}(v_{22})=1$. Let $G_2=(V_2, E_2)=G_0\setminus\{v_{11}, v_{22}\}$. Then $|V_2|=8$, $|E_2|=8$ and $s_{G_2}(v_{21})=3$. Consequently, $g(G_2)\leqq 7$. $\blacksquare$

Proposition 2. *If* $g(G)\geqq 8$, *then* $h(\bar{G})\leqq 6$.

Proof. Let $G=(V, E)$, $\bar{G}=(V, \bar{E})$. Suppose that in $\bar{G}$ there is a sparse set of edges $\bar{E}'=\{\bar{e}_1, \ldots, \bar{e}_7\}$. Denote by H a spanning subgraph of the graph $\bar{G}$, generated by $\bar{E}'$. Let us investigate the properties of H. Their combination will disprove the existence of H.

(I). *Degrees in H do not exceed* 2.

Proof of (I). Suppose that the vertex v_0 of H is incident with three edges $\bar{e}_1=(v_0, v_1)$, $\bar{e}_2=(v_0, v_2)$ and $\bar{e}_3=(v_0, v_3)$ of $\bar{E}'$. Since $\bar{E}'$ is sparse, then $\{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}\subset E$ which contradicts to the condition $g(G)\geqq 8$. $\blacksquare$

(II). *Every maximal matching in H contains at most* 4 *edges.*

Proof of (II). Suppose that in H there is a matching with edges $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_5$ where $\bar{e}_i=(v_{1i}, v_{2i})$ $(i=1, 2, \ldots, 5)$. Denote by $G_0=(V_0, E_0)$ an induced subgraph of G on the set $V_0=\bigcup_{i=1}^{5}\{v_{1i}, v_{2i}\}$. Due to Lemma 2, $G_0\cong C_{10}$. Let $\bar{e}_6=(v_{16}, v_{26})$.

Case 1. $\{v_{16}, v_{26}\} \cap V_0 = \emptyset$. Since $\bar{E}'$ is sparse then $\{v_{16}, v_{26}\}$ is connected with V_0 by at least 5 edges. Consequently, either v_{16} or v_{26} is adjacent to at least 3 vertices of $G_0 \cong C_{10}$. But then $g(G) \leqq 5$.

Case 2. $v_{26} \notin V_0, v_{16} = v_{11}$. Due to Lemma 2, an induced subgraph of the graph G on the set $(V_0 \setminus \{v_{21}\}) \cup \{v_{26}\}$ is also isomorphic to C_{10}. But then v_{26} is adjacent to the same vertices of V_0 as v_{11}. That is, there is a 4-cycle in G.

Case 3. $v_{16} = v_{11}$, $v_{26} = v_{12}$. Then $\{(v_{11}, v_{22}), (v_{21}, v_{12})\} \subset E_0$ and $|E_0| \geqq 11$. Contradiction to Lemma 2. ▮

(III). *In H there are no k-cycles for $k \neq 4$.*

Proof of (III). By definition, there are no 3-cycles in H. Since $|\bar{E}'| = 7$ then in H there are no k-cycles for $k \geqq 8$. If in H there is a cycle $(v_1, v_2, \ldots, v_k)$ then $(v_i, v_{i+2}) \in E$ for all $1 \leqq i \leqq k$. That is, for $5 \leqq k \leqq 7$ in G there is a cycle on these k vertices. ▮

(IV). *If in H there are two chains of length 2, (v_1, v_2, v_3) and (v_4, v_5, v_6) $(v_i \neq v_j$ for $i \neq j)$ and $(v_2, v_5) \notin E$, then*

a) the sets $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ are connected in G by exactly three edges,

b) these three edges form a matching.

Proof of (IV). Let the conditions of (IV) be satisfied. Then $\{(v_1, v_3), (v_4, v_6)\} \subset E$. Since there are no 3-cycles in G, then v_2 cannot be adjacent with v_4 and v_6 simultaneously.

Case 1. $(v_2, v_4) \notin E, (v_2, v_6) \notin E$. Since there are no 3-cycles in G then $|\{(v_5, v_1), (v_5, v_3)\} \cap E| \leqq 1$. Let, for definiteness, $(v_1, v_5) \notin E$. Then, for $(v_1, v_2) \in \bar{E}'$ to be adjacent in G with $(v_4, v_5) \in \bar{E}'$ and $(v_5, v_6) \in \bar{E}'$, it is necessary that $(v_1, v_4) \in E$ and $(v_1, v_6) \in E$. That is, in G there is a 3-cycle (v_1, v_4, v_6). Consequently, Case 1 is impossible.

Case 2. $(v_2, v_4) \in E, (v_2, v_6) \notin E$. Due to the symmetry of the chains (v_1, v_2, v_3) and (v_4, v_5, v_6) and the impossibility of Case 1, v_5 is adjacent with either v_1 or v_3. Let $(v_1, v_5) \in E$. Then $(v_3, v_5) \notin E$. Since $(v_2, v_3) \in \bar{E}'$ must be adjacent with $(v_5, v_6) \in \bar{E}'$ in G then $(v_3, v_6) \in E$. ▮

(V). *There are no 4-cycles in H.*

Proof of (V). Suppose that in H there is a cycle (v_1, v_2, v_3, v_4) where $\bar{e}_i = (v_i, v_{i+1}) \in \bar{E}'$ $(i = 1, 2, 3, 4)$. Then $(v_1, v_3) \in E$, $(v_2, v_4) \in E$.

If the set $(\bar{e}_5, \bar{e}_6, \bar{e}_7)$ is a matching then in H there is a matching $\bar{e}_1, \bar{e}_3, \bar{e}_5, \bar{e}_6, \bar{e}_7$, which contradicts (II).

Let $\bar{e}_5 = (v_5, v_6)$, $\bar{e}_6 = (v_6, v_7)$. Then $(v_5, v_7) \in E$. Applying (IV) to the chains (v_1, v_2, v_3) and (v_5, v_6, v_7), we obtain that v_6 is adjacent in G with at least one of the vertices $\{v_1, v_2, v_3\}$. Let, for definiteness, $(v_6, v_1) \in E$. Then $(v_6, v_3) \notin E$. If we apply (IV) to the chains (v_2, v_3, v_4) and (v_5, v_6, v_7) then the sets $\{v_2, v_3, v_4)$ and $\{v_5, v_6, v_7\}$ are connected by 3 edges. So, on the whole we have at least 7 edges on $\{v_1, v_2, \ldots, v_7\}$. ▮

(VI). *In H there are no three independent chains of length 2.*

Proof of (VI). Suppose that in H there are three nonintersecting chains (v_1, v_2, v_3), (v_4, v_5, v_6) and (v_7, v_8, v_9). Then $\{(v_1, v_3), (v_4, v_6), (v_7, v_9)\} \subset E$. Since in G there are no 3-cycles, then at least one of the edges (v_2, v_5), (v_5, v_8), (v_8, v_2) does not belong to E. Let $(v_2, v_5) \notin E$. According to (IV), the sets $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6\}$ are connected in G by three edges exactly. If $\{(v_2, v_8), (v_5, v_8)\} \subset E$ then there are seven edges on 7 vertices $V_1 \cup V_2 \cup \{v_8\}$. Let $(v_2, v_8) \notin E$. Then V_1 is joined to $V_3 = \{v_7, v_8, v_9\}$ in G, due to (IV), by exactly 3 edges. Since V_2 is connected with V_3 by at least one edge, then on 9 vertices $V_1 \cup V_2 \cup V_3$ we have at least 10 edges of G and, due to Lemma 1, in G there is a cycle of length $\leqq 6$. ▌

(VII). *The length of any chain in H is at most* 5.

Proof of (VII). Suppose that in H there is a chain $(v_1, v_2, \ldots, v_7)$. Then $E_0 = \{(v_1, v_3), (v_2, v_4), (v_3, v_5), (v_4, v_6), (v_5, v_7)\} \subset E$. Since in G there is no 3-cycle (v_2, v_4, v_6) then $(v_2, v_6) \notin E$. Applying (IV) to the chains (v_1, v_2, v_3) and (v_5, v_6, v_7), we obtain that in G the sets $\{v_1, v_2, v_3\}$ and $\{v_5, v_6, v_7\}$ are connected by at least two edges which do not belong to E_0. Thus, on 7 vertices $v_1, v_2, \ldots, v_7$ in G there are $\geqq 7$ edges. ▌

(VIII). *If in H there is a chain of length 4 then there is no chain of length* 2 *which is disjoint with it.*

Proof of (VIII). Suppose that in H there are disjoint chains $(v_1, v_2, \ldots, v_5)$ and (v_6, v_7, v_8). Then $\{(v_1, v_3), (v_2, v_4), (v_3, v_5), (v_6, v_8)\} \subset E$. Since in G there are no 3-cycles, then v_7 is adjacent in G neither to v_2 nor to v_4.

Case 1. $(v_7, v_2) \notin E$, $(v_7, v_4) \notin E$. Applying (IV) to pairs of chains (v_1, v_2, v_3), (v_6, v_7, v_8) and (v_3, v_4, v_5), (v_6, v_7, v_8), we obtain that v_2 is adjacent to either v_6 or v_8, and v_4 is adjacent to either v_6 or v_8. That is, in G there is a cycle of the length $\leqq 4$.

Case 2. $(v_7, v_2) \in E$, $(v_7, v_4) \notin E$. Applying (IV) to the chains (v_3, v_4, v_5) and (v_6, v_7, v_8), we obtain that $\{v_3, v_4, v_5\}$ and $\{v_6, v_7, v_8\}$ are connected in G by exactly three edges. Consequently, in the subgraph of the graph G on the vertices $v_2, v_3, \ldots, v_8$ there are at least 7 edges. ▌

Completion of the proof of Proposition 2. What can H be, satisfying properties (I)—(VIII)? According to (I), (III), (V), only chains can be connectedness components of H. According to (II) their number does not exceed 4.

Consider all possible cases ($k(H)$ is the number of connectedness components of H).

Case 1. $k(H) = 4$. Due to (II) the length of every chain does not exceed two. Consequently, the only possible collection of chain lengths is 2, 2, 2, 1. But it is also impossible in view of (VI).

Case 2. $k(H) = 3$. According to (VI) at least one of the chains has length 1. So, possible collections of chain lengths are: a) 5, 1, 1; b) 4, 2, 1; c) 3, 3, 1. But a) and c) contradict (II), and the subcase b) contradicts (VIII).

Case 3. $k(H) = 2$. Possible collections of chain lengths are: a) 6, 1; b) 5, 2; c) 4, 3. The subcase a) contradicts (VII), and b) and c) contradict (VIII).

Case 4. $k(H)=1$. This is impossible in view of (VII). ▮

Remark 1. If $G \cong C_7$, then $h(\bar{G})=7$.

Remark 2. Similar to the proof of Proposition 2, one can show that if $g(G) \geqq 6$ then $h(\bar{G}) \leqq 7$. There exists a graph G_0 such that $g(G_0)=5$ and $h(G_0)=8$.

4. The upper bound for t_0

We remind the known result

Theorem 3 (Erdős [4]). *If l and η are fixed and $0<\eta<1/(2l)$, then for sufficiently large n there is an n-vertex graph G containing no cycles of length l and with $\chi(G) \geqq n^\eta$.* ▮

From Proposition 2 and Theorem 3 we obtain at once

Proposition 4. $c_6 > 1/15$. ▮

Thus, if $h(\bar{G}) \leqq 5$, then $\chi(G) \leqq h(\bar{G})$. But if $h(\bar{G})=6$ then $\chi(G)$ can be arbitrarily large.

Remark 3. From the validity of Remark 2 and from Theorem 3 it easily follows that $c_7 > 1/11$. The bound for c_7 seems to be improvable. Bounds for c_t at $t \geqq 7$ can be obtained combining the ideas of Section 3 and the probabilistic methods.

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References

[1] R. C. Brigham and R. D. Dutton, On clique covers and independence numbers of graphs, *Discrete Math.*, **44** (1983), 139—144.

[2] P. Erdős, On the covering of the vertices of a graph by cliques, *J. Math. Res. Exposition*, **2** (1982), 93—96.

[3] P. Erdős, A. W. Goodman and L. Pósa, The representations of a graph by set intersections, *Canad. J. Math.*, **18** (1966), 106—112.

[4] P. Erdős and J. Spencer, *Probabilistic methods in combinatorics*, Akadémiai Kiadó, Budapest, 1974, (Theorem 11.2).

[5] J. Orlin, Contentement in Graph Theory, *K. Nederl. Akad. Wetensch. Proc. (Ser. A)*, **80** (1977), 406—424.

[6] K. R. Parthasaraty and S. A. Choudum, The edges clique cover number of products of graphs, *Journ. Math. Phys. Sci.*, **10** (1976), 3, 255—261

A. V. Kostochka

Institute of Mathematics
of the Siberian Branch of the
Academy of Sciences of the USSR,
Novosibirsk-90, 630090, USSR